

# Directed transport driven by Lévy flights coexisting with subdiffusion

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## Abstract

Transport of the Brownian particles driven by Lévy flights coexisting with subdiffusion in asymmetric periodic potentials is investigated in the absence of any external driving forces. Using the Langevin-type dynamics with subordination techniques, we obtain the group velocity which can measure the transport. It is found that the group velocity increases monotonically with the subdiffusive index and there exists an optimal value of the Lévy index at which the group velocity takes its maximal value. There is a threshold value of the subdiffusive index below which the ratchet effects will disappear. The nonthermal character of the Lévy flights and the asymmetry of the potential are necessary to obtain the directed transport. Some peculiar phenomena induced by the competition between Lévy flights and subdiffusion are also observed. The pseudo-normal diffusion will appear on the level of the median.

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## I. INTRODUCTION

Directed Brownian motion induced by zero-mean non-equilibrium fluctuations in the absence of macroscopic forces and potential gradients is presently under intense investigation<sup>1</sup>. This comes from the desire to understand unidirectional transport in biological systems<sup>2</sup>, as well as their potential technological applications ranging from classical non-equilibrium models<sup>3</sup> to quantum systems<sup>4</sup>. A ratchet system is generally defined as a system that is able to transport particles in a periodic structure with nonzero macroscopic velocity in the absence of macroscopic force on average. Broadly speaking, ratchet devices fall into three categories depending on how the applied perturbation couples to the substrate asymmetry: rocking ratchets<sup>5</sup>, flashing ratchets<sup>6</sup>, and correlation ratchets<sup>7</sup>. Additionally, entropic ratchets, in which Brownian particles move in a confined structure, instead of a potential, were also extensively investigated<sup>8</sup>. These studies on directed transport of the Brownian particles focused on the normal diffusion.

However, anomalous diffusion has attracted growing attention, being observed in various fields of physics and related sciences<sup>9</sup>. Description of physical models in terms of Lévy flights and subdiffusion becomes more and more popular<sup>10–23</sup>. In the complex systems the distinct class of subdiffusion processes was reported in condensed phases<sup>10</sup>, ecology<sup>11</sup>, and biology<sup>12</sup>. Superdiffusion driven by Levy flights is actually observed in various real systems and is used to model a variety of processes such as bulk mediated surface diffusion<sup>13</sup>, exciton and charge transport in polymers under conformational motion<sup>14</sup>, transport in micelle systems or heterogeneous rocks<sup>15</sup>, two-dimensional rotating flow<sup>16</sup>, and many others<sup>9</sup>. Goychuk and coworkers<sup>17</sup> studied the subdiffusive transport in tilted periodic potentials and established a universal scaling relation for diffusive transport. Dybiec and coworkers<sup>18</sup> studied

the minimal setup for a Lévy ratchet and found that due to the nonthermal character of the Lévy noise, the net current can be obtained even in the absence of whatever additional time-dependent forces. Del-Castillo-Negrete and coworkers<sup>19</sup> also found the similar results in constant force-driven Lévy ratchet. Rosa and Beims<sup>20</sup> studied the optimal transport and its relation to superdiffusive transport and Lévy walks for Brownian particles in ratchet potential in the presence of modulated environment and external oscillating forces. We also studied the transport of Brownian particles in the presence of ac-driving forces and Lévy flights and multiple current reversals were observed<sup>21</sup>

Recently, much attention has been devoted to the competition between subdiffusion and Lévy flights. The competition is conveniently described by the fractional Fokker-Planck equation with temporal and spatial fractional derivatives<sup>9</sup>. It is very difficult to see this competition in the framework of the fractional Fokker-Planck dynamics. Magdziarz and coworkers<sup>22</sup> proposed a equivalent approach based on the subordinated Langevin method to visualize the competition on the level of sample paths as well as on the level of probability density functions. Based on this approach Dybiec and coworkers<sup>23</sup> found that due to the competition between Lévy flights and subdiffusion, the standard measure used to discriminate between anomalous and normal behavior cannot be applied straightforwardly. Koren and coworkers<sup>24</sup> have investigated the first passage times in one-dimensional system displaying a competition between subdiffusion and Lévy flights and found some peculiar phenomena.

What will happen when the particles move in a ratchet potential subjected to subdiffusion and Lévy flights? In order to answer this question we use the subordinated Langevin method proposed by the Magdziarz and coworkers<sup>22</sup> to investigate this competition in a minimal Lévy

ratchet without any external driving forces. We emphasize on visualizing the competition on the level of the group velocity and diffusion and finding how this competition affects the directed transport.

## II. MODEL AND METHODS

We consider the transport of the Brownian particles driven by Lévy flights and subdiffusion in the absence of whatever additional time-dependent forces. The competition between Lévy flights and subdiffusion in a ratchet potential  $V(x)$  can be described by the fractional Fokker-Planck equation with temporal and spatial fractional derivatives<sup>9,22</sup>

$$\frac{\partial p(x, t)}{\partial t} = {}_0D_t^{1-\alpha} \left[ \frac{\partial}{\partial x} \frac{V'(x)}{\eta} + D \frac{\partial^\mu}{\partial |x|^\mu} \right] p(x, t), \quad (1)$$

where  $p(x, t)$  is the probability density for particles at position  $x$  and time  $t$ . The prime stands for the derivative with respect to the space variable  $x$ .  $D$  is the anomalous diffusion coefficient which describes the noise intensity in the subordinated process.  $\eta$  denotes the generalized friction constant. Here  ${}_0D_t^{1-\alpha}$  is the fractional of the Riemann-Liouville operator ( $0 < \alpha \leq 1$ ) defined through<sup>9,22</sup>

$${}_0D_t^{1-\alpha} g(t) = \frac{1}{\Gamma(\alpha)} \frac{d}{dt} \int_0^t (t-s)^{\alpha-1} g(s) ds, \quad (2)$$

where  $\Gamma(\cdot)$  is the Gamma function. From the definition, it becomes apparent that subdiffusion corresponds to a slowly decaying memory integral in the dynamical equation for  $p(x, t)$ . The operator  $\frac{\partial^\mu}{\partial |x|^\mu}$ ,  $0 < \mu \leq 2$ , stands for the Riesz fractional derivative<sup>9,22</sup> with the Fourier transform  $\mathcal{F}\left\{\frac{\partial^\mu}{\partial |x|^\mu} f(x)\right\} = -|k|^\mu \tilde{f}(k)$ .

The occurrences of the operator  ${}_0D_t^{1-\alpha}$  and  $\frac{\partial^\mu}{\partial |x|^\mu}$  are induced by the heavy-tailed waiting times between successive jumps and the heavy-tailed distributions of the jumps, respectively,

in the underlying continuous-time random walk scheme. The case of  $\alpha = 1$ ,  $\mu = 2$  corresponds to the standard Fokker-Planck equation.  $V(x)$  is an asymmetric periodic potential

$$V(x) = \frac{V_0}{2\pi} \left[ \sin(2\pi x) + \frac{\Delta}{4} \sin(4\pi x) \right], \quad (3)$$

where  $V_0$  and  $\Delta$  are the amplitude and the asymmetric parameter of the potential, respectively.

Because it is very difficult to solve Eq. (1) analytically and numerically, we used the subordinated Langevin method proposed by Magdziarz and coworkers<sup>22</sup> to investigate the transport. In their method, the solution  $p(x, t)$  of Eq. (1) is equal to the probability density function of the subordinated process

$$Y(t) = X(S_t), \quad (4)$$

where the parent process  $X(\tau)$  is defined as the solution the stochastic differential equation

$$dX(\tau) = -\frac{V'(X(\tau))}{\eta} d\tau + D^{1/\mu} dL_\mu(\tau), \quad (5)$$

where  $L_\mu(\tau)$  is the symmetric  $\mu$ -stable Lévy motion with the Fourier transform  $\mathcal{F}\{L_\mu(\tau)\} = e^{-\tau|k|^\mu}$ . Employing the Euler scheme to Eq. (5), one can obtain

$$X(\tau_0) = 0, \quad (6)$$

$$X(\tau_i) = X(\tau_{i-1}) - \frac{V'(X(\tau_{i-1}))}{\eta} \Delta\tau + (D\Delta\tau)^{1/\mu} \xi_i, \quad (7)$$

where  $i = 1, 2, 3, \dots$  and  $\xi_i$  are the random variables with standard symmetric  $\mu$ -stable distribution. The procedure of generating realizations  $\xi_i$  is the following<sup>22,25</sup>

$$\xi_i = \frac{\sin(\mu V)}{(\cos V)^{1/\mu}} \left[ \frac{\cos([1 - \mu]V)}{W} \right]^{\frac{1-\mu}{\mu}}, \quad (8)$$

where the random variable  $V$  is uniformly distributed on  $(-\pi/2, \pi/2)$ ,  $W$  has exponential distribution with mean one.

The inverse-time  $\alpha$ -stable subordinator  $S_t$ , which is assumed to be independent of  $X(\tau)$ , is defined as

$$S_t = \inf\{\tau : U(\tau) > t\}, \quad (9)$$

where  $U(\tau)$  is the strictly increasing  $\alpha$ -stable Lévy motion with Laplace transform  $\mathcal{L}\{U(\tau)\} = e^{-\tau k^\alpha}$ .

Using the standard method of summing increments of the process  $U(\tau)$  one can get

$$U(\tau_0) = 0, \quad (10)$$

$$U(\tau_j) = U(\tau_{j-1}) + \Delta\tau^{1/\alpha}\zeta_j, \quad (11)$$

where  $j = 1, 2, 3, \dots$  and  $\zeta_j$  are the skewed positive  $\alpha$ -stable random variables<sup>22,25</sup>. The method to generate the random variables is

$$\zeta_j = \frac{\sin(\alpha(V + \frac{\pi}{2}))}{[\cos(V)]^{\frac{1}{\alpha}}} \left[ \frac{\cos(V - \alpha(V + \frac{\pi}{2}))}{W} \right]^{\frac{1-\alpha}{\alpha}}, \quad (12)$$

where  $V$  and  $W$  have the same definitions as that in Eq. (8). From the above procedures, one can obtain the subordinated process  $Y(t)$  and its probability distribution function is equal to the solution of Eq. (1). For more detailed information on the algorithm, please see the Ref. (22).

In the classical ratchets, one can use the average velocity and effective diffusion coefficient to describe the transport. However, for the noise with distribution of a Lévy-stable law, the mean of the noise and the second moment may do not exist. As a consequence, the classical stochastic theory (average velocity and effective diffusion coefficient), which is based on the ordinary central limit theorem, is no longer valid. To overcome this problem, Dybiec and

coworkers<sup>18</sup> recently proposed a different approach to the Lévy ratchet problem based on the quantile line analysis for  $0 < \mu < 2$ .

Quantile line is a very useful tool for investigation of the overall motion of the probability density of finding a particle in the vicinity of  $Y(t)$ <sup>18,23</sup>. A median line for a stochastic process  $Y(t)$  is a function of  $q_{0.5}(t)$  given by the relationship  $Pr(Y(t) \leq q_{0.5}(t)) = 0.5$ . Therefore, one can use the derivative of the median to define the group velocity of the particle packet<sup>18</sup>,

$$V_g = \frac{dq_{0.5}(t)}{dt}, \quad (13)$$

and this definition is valid even for the case of lacking average velocity.

### III. NUMERICAL RESULTS AND DISCUSSION

In order to investigate the competition between Lévy flights and subdiffusion in a ratchet potential, we carried out extensively numerical simulations based on the subordinated Langevin method<sup>22</sup>. For simplicity we set  $\eta = 1.0$  and  $1 < \mu \leq 2$  throughout the work. In our simulations, we have considered more than  $10^5$  realizations to obtain the accurate median. In order to provide the requested accuracy of the system the dynamics time step was chosen to be smaller than  $10^{-3}$ . We have checked that these are sufficient for the system to obtain consistent results.

Firstly, we will investigate the diffusive properties of the Brownian particles. Usually, the types of the diffusion processes are determined by the spread of the distance traveled by a random walker. The diffusion is characterized through the power law form of the mean-square displacement  $\langle x^2(t) \rangle \propto t^\delta$ . According to the value of the index  $\delta$ , one can distinguish subdiffusion ( $0 < \delta < 1$ ), normal diffusion ( $\delta = 1$ ) and superdiffusion ( $\delta > 1$ ). Here, we use the median of square displacement  $M(x^2)$ , instead of mean-square displacement, to

characterize the diffusion. Fig. 1 (a) shows the time dependence of  $M(x^2)/t$  for different combinations of  $\mu$  and  $\alpha$  without any external potential. It is found that the linear time dependence of the median of square displacement,  $M(x^2) \propto t$ , will occur for the case of  $\frac{2\alpha}{\mu}=1$ , which indicates the normal diffusion. However, this is not true, for example  $\mu = 1.8$  and  $\alpha = 0.9$ , the process is still non-Markov and non-Gaussian. This pseudo-normal diffusion is due to the competition between Lévy flights and subdiffusion. Dybiec and coworkers<sup>23</sup> have presented discussions in detail on this paradoxical diffusion. Fig. 1 (b) presents the time dependence of  $M(x^2)/t$  in the presence of a ratchet potential. Interestingly, the pseudo-normal diffusion for  $\mu = 1.8$  and  $\alpha = 0.9$  with external potentials is not normal.

Next, we will study the rectified mechanism of the Lévy ratchets. Usually, the ratchet mechanism demands three key ingredients<sup>26</sup> which are (a) nonlinear periodic potential: it is necessary since the system will produce a zero mean output from zero-mean input in a linear system; (b) asymmetry of the potential, it can violate the symmetry of the response; (c) fluctuating: Lévy flights can break thermodynamical equilibrium. In Fig. 2 (a), we studied the time dependence of the median for different values of the asymmetry parameter  $\Delta$  at  $\mu = 1.5$  and  $\alpha = 1.0$ . The median is positive for  $\Delta > 0$ , zero at  $\Delta = 0$ , and negative for  $\Delta < 0$ . Therefore, the asymmetry of the potential will determine the direction of the transport and no directed transport occurs in a symmetric potential. Now we will give the physical interpretation of the directed transport for the case of  $\Delta = 1$ . Firstly, the particles stay in the minima of the potential awaiting large noise pulse to be catapulted out. The particles will be thrown out to the left and the right with the equal probabilities. In this case, the distance from minima to maxima is shorter from the right side than that from the left side. Consequently, most of the particles are thrown out from the right side, resulting

in positive transport. This gives rise to the overall preferred motion to the right.

Figure 2 (b) gives the time dependence of the median for different combinations of  $\mu$  and  $\alpha$ . We find that Lévy flights are necessary to obtain the directed transport. For Gaussian case ( $\mu = 2.0$ ), directed transport disappears. This is due to the nonthermal character of the Lévy flights that can break thermodynamical equilibrium. From Fig. 2(a) and (b) we can see that the asymmetry of the potential and the non-equilibrium character of the Lévy flights are the two necessary conditions for directed transport. The direction of the transport is determined by the direction of the steeper slope of the potential and the Lévy flights can break thermodynamical equilibrium. These two key ingredients can realize the ratchet effects.

Figure 3 illustrates the dependence of the group velocity  $V_g$  on the subdiffusive index  $\alpha$  for different values of the Lévy index  $\mu$ . One can see that group velocity  $V_g$  increases monotonically with the subdiffusive index  $\alpha$ . For small values of  $\alpha$ , the waiting time between successive jumps is very long and it is not easy for particles to pass across the barrier. Thus, most particles will stay in their original minima of the potential and the group velocity becomes very small. Especially, we also find that there exists a threshold value of  $\alpha$  below which no directed transport can be obtained. The subdiffusion dominates the transport for small values of  $\alpha$  ( $\alpha < 0.7$ ), while the effects of the Lévy flights become preponderant for large values of  $\alpha$ .

Figure 4 shows the dependence of the group velocity  $V_g$  on the Lévy index  $\mu$  for different values of  $\alpha$ . When  $\mu \rightarrow 2.0$ , the system is under thermodynamical equilibrium and no directed transport appears. For small values of  $\mu$ , Lévy flights are longer and the outliers in the Lévy noise are larger. In this case, the effects of the asymmetry of the potential become

very small, resulting in small group velocity. Therefore, there exists a optimal value of  $\mu$  at which the group velocity takes its maxima. This can also be confirmed by Fig. 3. For very small values of  $\alpha$ , for example  $\alpha = 0.5$ , the group velocity is zero for all values of  $\mu$  and the transport is absolutely dominated by subdiffusion.

The group velocity  $V_g$  as a function of noise intensity  $D$  is shown in Fig. 5 for different combinations of  $\mu$  and  $\alpha$ . The curve is observed to be bell shaped which shows the feature of resonance. When  $D \rightarrow 0$ , the particles cannot pass across the barrier and there is no directed current. When  $D \rightarrow \infty$  so that the noise is very large, the effect of the potential disappears and the group velocity tends to zero, also. There is an optimal value of  $D$  at which the group velocity is maximal. There are two intersections ( $D_{c1}$  and  $D_{c2}$ ) between the line of  $\mu = 1.9$  and  $\alpha = 1.0$  and the line of  $\mu = 1.5$  and  $\alpha = 0.9$ . For simplicity, we define  $V_g(\mu, \alpha)$  as the group velocity for different values of  $\mu$  and  $\alpha$ . When  $D < D_{c1}$ ,  $V_g(1.9, 1.0) > V_g(1.5, 0.9)$ , Lévy flights dominates the transport. When  $D_{c1} < D < D_{c2}$ ,  $V_g(1.9, 1.0) < V_g(1.5, 0.9)$ , the transport is governed by subdiffusion. For the case of  $D > D_{c2}$ , all particles can easily pass across the barrier and Lévy flights will mainly contribute to the transport and  $V_g(1.9, 1.0) > V_g(1.5, 0.9)$ .

In Fig. 6, we plot the dependence of the group velocity  $V_g$  on the amplitude  $V_0$  of the potential for different combinations of  $\mu$  and  $\alpha$ . When  $V_0 \rightarrow 0$ , the effects of the potential disappear and the group velocity tends to zero. When  $V_0 \rightarrow \infty$ , the particles cannot pass across the barrier and the group velocity goes to zero, also. Thus, the curve shows a peak. Remarkably, there is an intersection between the line of  $\mu = 1.5$  and  $\alpha = 0.9$  and the line of  $\mu = 1.9$  and  $\alpha = 1.0$ . This is due to the competition between Lévy flights and subdiffusion. When  $V_0 < V_c$ , Lévy flights are predominant and  $V_g(1.5, 0.9) > V_g(1.9, 1.0)$ . When  $V_0 > V_c$ ,

subdiffusion dominates the transport and  $V_g(1.5, 0.9) < V_g(1.9, 1.0)$ . In this case, the height of the barrier is very high and few particles driven by Lévy flights can pass across the barrier, the effects of the Lévy flights will disappear and subdiffusion will play a major role.

#### IV. CONCLUDING REMARKS

In this paper, we have investigated the directed transport of the Brownian particles in a ratchet potential driven by Lévy flights coexisting with subdiffusion. We used recently developed framework of Monte Carlo simulation<sup>22</sup> which is equal to the solution of the fractional Fokker-Planck equation. The group velocity proposed by Dybiec and coworkers<sup>18</sup> is used to measure the transport. It is found that the group velocity increases monotonically with the subdiffusive index, while the group velocity as a function of the Lévy index is nonmonotonic. The former is caused by the increase of the waiting time between successive jumps and the latter is owing to the interplay between Lévy flights and the height of the barriers. There is a threshold value of  $\alpha$  below which the transport is absolutely dominated by subdiffusion and the directed transport disappears. The dependences of the group velocity on the noise intensity and the amplitude of the potential are also investigated. There is an optimal value of the noise intensity (the amplitude of the potential) at which the group velocity is maximal. The competition between Levy flights and subdiffusion in the ratchet potential is observed on the level of the group velocity as well as the median of square displacement. The nonthermal character of the Lévy flights and the asymmetry of the potential are the necessary conditions for directed transport when the system is in the absence of any external driving forces. Because of this competition, we also found the pseudo-normal diffusion reported by Dybiec and coworkers<sup>23</sup>, in which time dependence of

the median of square displacement is linear,  $M(x^2) \propto t$ , while the process is still non-Markov and non-Gaussian.

Anomalous transport is becoming widely recognized in a variety of the fields. Beyond its intrinsic theoretical interest, the results we have presented may have wide applications in some complex systems, such as diffusive transport in plasmas, particles separation with non-Gaussian diffusion, and ratchet transport in biology systems that are intrinsically out of equilibrium.

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## V. CAPTION LIST

Fig. 1. Time dependence of  $M(x^2)/t$  for different combinations of  $\mu$  and  $\alpha$ : (a) without external potential at  $D = 0.4$ , solid lines present  $t^{\frac{2\alpha}{\mu}-1}$  scaling; (b) with external potential at  $D = 0.4$ ,  $V_0 = 5.0$ , and  $\Delta = 1.0$ .

Fig. 2. Time dependence of the median: (a) for different values of the asymmetry parameter  $\Delta$  at  $D = 0.4$ ,  $V_0 = 5.0$ ,  $\mu = 1.5$ , and  $\alpha = 1.0$ , the inset shows the potential profile; (b) for different combinations of  $\mu$  and  $\alpha$  at  $D = 0.4$ ,  $V_0 = 5.0$ , and  $\Delta = 1.0$ .

Fig. 3. Group velocity  $V_g$  versus subdiffusive index  $\alpha$  for different values of  $\mu$  at  $D = 0.4$ ,  $V_0 = 5.0$ , and  $\Delta = 1.0$ .

Fig. 4. Group velocity  $V_g$  versus Lévy index  $\mu$  for different values of  $\alpha$  at  $D = 0.4$ ,  $V_0 = 5.0$ , and  $\Delta = 1.0$ .

Fig. 5. Group velocity  $V_g$  as a function of noise intensity  $D$  for different combinations of  $\mu$  and  $\alpha$  at  $V_0 = 5.0$  and  $\Delta = 1.0$ .

Fig. 6. Group velocity  $V_g$  versus the amplitude  $V_0$  of the potential for different combinations of  $\mu$  and  $\alpha$  at  $D = 0.4$  and  $\Delta = 1.0$ .

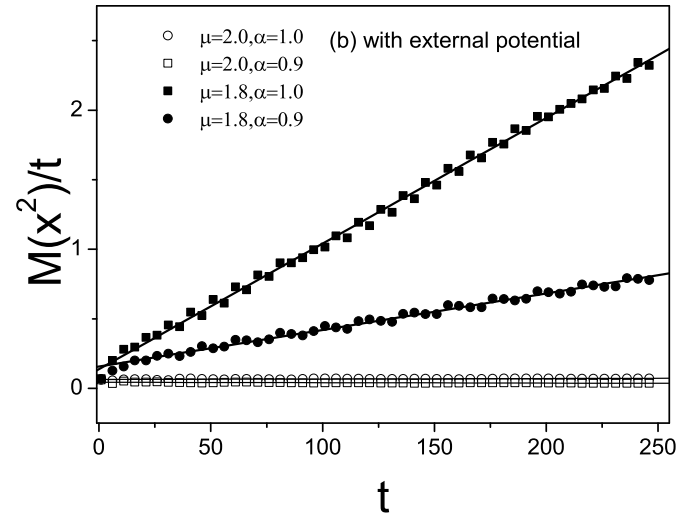
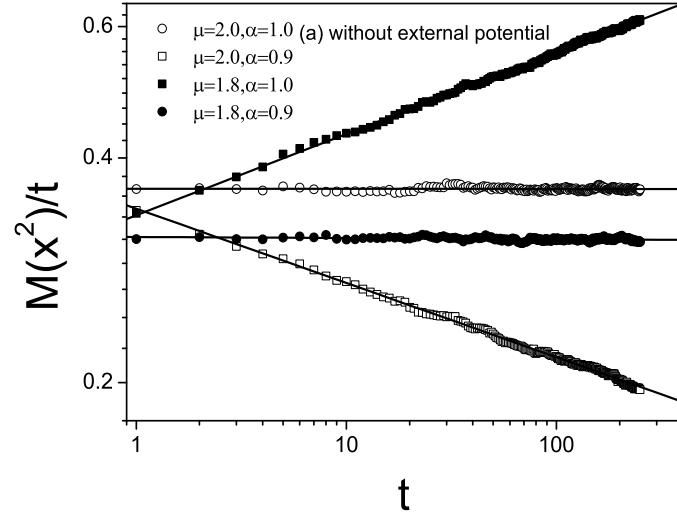


FIG. 1:

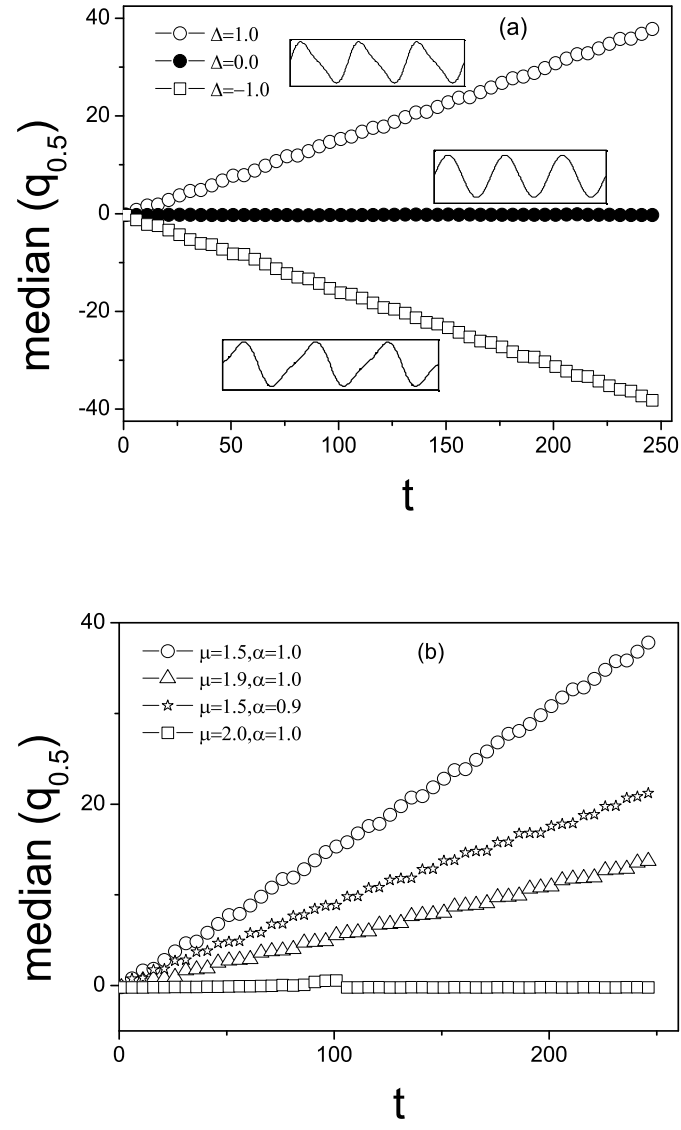


FIG. 2:

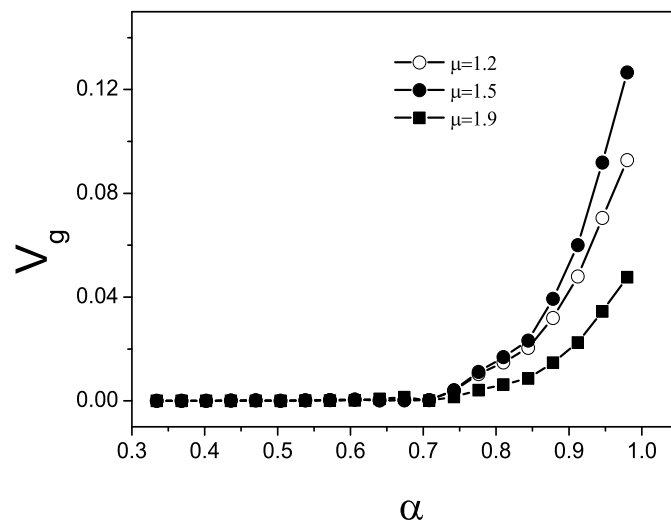


FIG. 3:

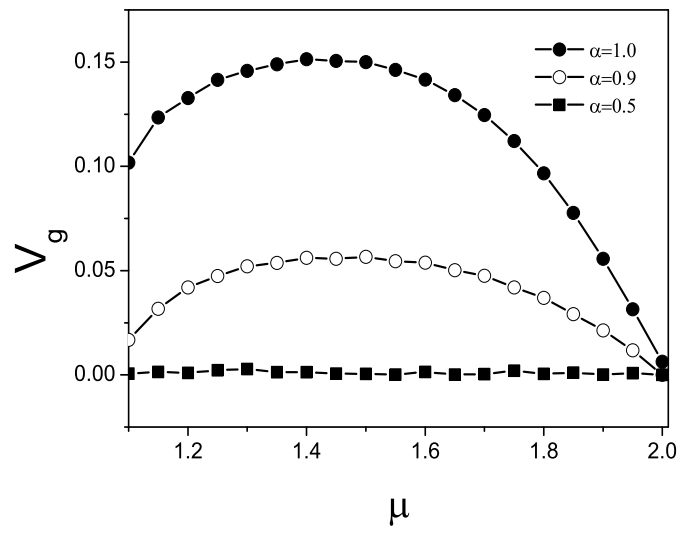


FIG. 4:

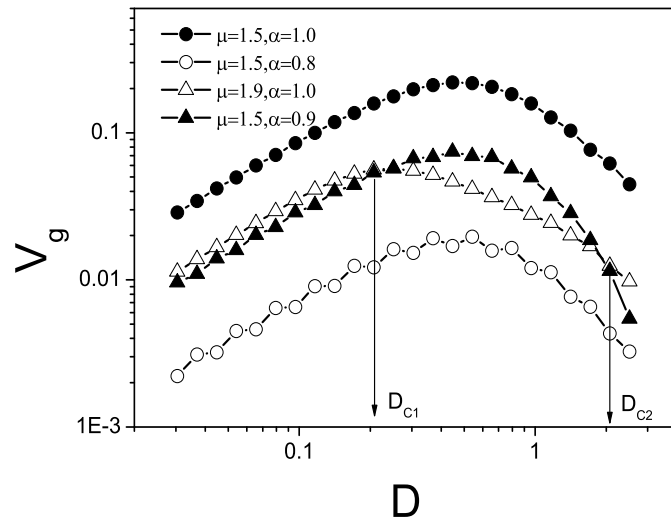


FIG. 5:

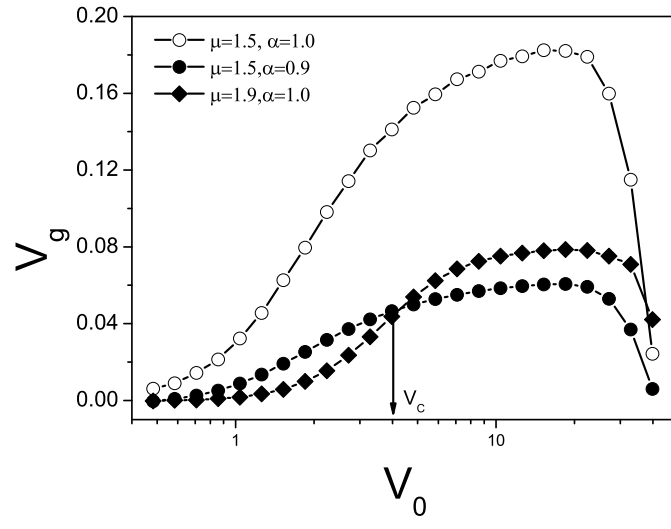


FIG. 6: